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THE FUNDAMENTAL GROUP OF THE COMPLEMENT FOR KLEIN'S ARRANGEMENT OF TWENTY-ONE LINES

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The classical Klein arrangement on the projective plane is derived from the three-dimensional unitary reflection group $(\mathbb{Z}/2\mathbb{Z})\times \mathrm{PSL}(2, F_7)$ which is of neither Coxeter nor Shephard type. In this paper a natural description by generators and relations is given to the fundamental group of the complement of the arrangement, and also to its extended group by $\mathrm{PSL}(2, F_7)$. The method is based on the birational geometry over the quotient of the plane and on the Zariski-van Kampen theory applied to the branch locus.

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van Kampen	unitary reflection group
$\mathrm{PSL}(2, F_7)$	fundamental group
Zariski	line arrangement
Klein	

Introduction

The group of order 168 of Klein is $\mathrm{PSL}(2, F_7) := \mathrm{SL}(2, F_7)/\{\pm I\}$ where as usual $\mathrm{SL}(2, F_7)$ denotes the group of unimodular (2, 2)-matrices in the finite field F_7 and I is the identity matrix. This group has a three-dimensional irreducible representation over $\mathbb{C}$, which is unique up to an automorphism; it induces the nontrivial action of the group over the projective plane $P_2(\mathbb{C})$. $\mathrm{PSL}(2, F_7)$ contains exactly twenty-one involutions, which are conjugate to each other. Since the group is simple, each involution has the eigenvalues $\{1, -1, -1\}$ on the representation space, which further implies that its fixed point set consists of a line and an isolated point with respect to the action over $P_2(\mathbb{C})$. In particular we obtain the twenty-one lines associated with the involutions, the union of which we called Klein's arrangement in the title. The purpose of this paper is to compute the fundamental group π_1 for the complement of the arrangement. By the existence of group action, this π_1 is naturally embedded into a larger group to be such a normal subgroup that the factor group is isomorphic to $\mathrm{PSL}(2, F_7)$. In fact this larger group is obtained by dividing by the center the fundamental group of the quotient of a suitable open subset of the complement over which PSL(2, F_7) acts freely. Using the invariant homogeneous polynomials of degree eighteen, this quotient is further described as the complement of a simple sextic curve (on the other $P_2(\mathbb{C})$), which enables us to compute explicitly π_1, by the usual method of Zariski and van Kampen. In the last section we will enjoy the beautiful structure of the fundamental group. This does not seem to be a coincidence at all. The main result is summarized in Theorem 3.7.

My interest in this topic was raised through a conversation with Peter Orlik during his stay (April 1988) at the Research Institute for Mathematical Sciences of the Kyoto University, for which I am very grateful to him. I also thank him for English revision and helpful comments.

1. Invariant polynomials and the quotient

Let V be the three-dimensional representation space of PSL(2, F_7) in the introduction. Following Klein [1], we take the coordinates (y_1, y_2, y_3) for V in such a way that the invariant homogeneous polynomial of the lowest degree 4 is given by

$$f := y_1^3 y_2 + y_2^3 y_3 + y_3^3 y_1 .$$

Then we obtain also the following important invariant polynomials:

$$\nabla := \operatorname{Hess}(f)/54 = 5y_1^2y_2^2y_3^2 - (y_1^5y_3 + y_2^5y_1 + y_3^5y_2),$$

$$C := \frac{1}{9}\begin{vmatrix} \partial^2 f/\partial y_i \partial y_j & \partial\nabla/\partial y_i \\ \partial\nabla/\partial y_j & 0 \end{vmatrix},$$

$$K := \frac{1}{14}\left|\frac{\partial(f, \nabla, C)}{\partial(y_1, y_2, y_3)}\right|.$$

Note that f, ∇, C are of the degrees 4, 6, 14 respectively and they are invariant under the action of the linear group generated by PSL(2, F_7) $\subseteq$ SL(V) and $-\mathrm{id}_V$. This subgroup of GL(V) is then generated by reflections and the ring of invariant polynomials for it is generated by f, ∇, C without any relation among them. Since K is of degree 21, K itself is not invariant under the action of this extended group, but K^2 is invariant. Thus K^2 is expressed to be a polynomial of f, ∇, C; explicitly we have [1]:

$$\begin{aligned} K^2 = {} & C^3 - 88\cdot f^2\nabla C^2 + 16\cdot 63\cdot f\nabla^4 C + 17\cdot 64\cdot f^4\nabla^2 C - 256\cdot f^7 C \\ & + 27\cdot 64\cdot \nabla^7 - 128\cdot 469\cdot f^3\nabla^5 + 43\cdot 512\cdot f^6\nabla^3 - 2048\cdot f^9\nabla. \end{aligned} \tag{1.1}$$

By $(f=0)$, $(\nabla=0)$ etc. we denote the invariant curves on $P_2(\mathbb{C})$: (y_1, y_2, y_3) defined by the equations $f=0$, $\nabla=0$ etc. $(K=0)$ is exactly Klein's arrangement of 21 lines.

Now we want to use these invariant polynomials to describe the quotient of $P_2(\mathbb{C})$: (y_1, y_2, y_3) by PSL(2, F_7). We first state that the quotient is again birationally $P_2(\mathbb{C})$.

Observe that $\{\nabla^3, fC, f^3\nabla\}$ is a base for the space of invariant homogeneous polynomials of degree 18. Note further that $\mathbb{C}[f, \nabla, C]$ and $\mathbb{C}[\nabla^3, fC, f^3\nabla]$ have the same quotient field, which implies that the mapping of $P_2(\mathbb{C})$: (y_1, y_2, y_3) to $P_2(\mathbb{C})$: (ξ, η, ζ) defined by

$$\xi = 4\nabla^3, \qquad \eta = fC, \qquad \zeta = 8f^3\nabla$$

gives the birational mapping of the quotient surface $S := P_2(\mathbb{C})/\mathrm{PSL}(2, F_7)$ onto $P_2(\mathbb{C})$: (ξ, η, ζ). This birational mapping becomes holomorphic, when lifted to the minimal desingularization $\hat{S}$ of S; so the birational morphism $\hat{S}$ to $P_2(\mathbb{C})$: (ξ, η, ζ) is a blowing down of $\hat{S}$. Next we find the singular points of S, which are all cyclic quotients. The group $\mathrm{PSL}(2, F_7)$ has exactly twenty-one cyclic groups of order 4, each containing exactly one involution and thus corresponding to the twenty-one involutions. Let H be a subgroup of order 4 and h a generator of H. Then h^2 is the involution contained in H. H has the three fixed points, one of which is the isolated fixed point of h^2. The image of this point in S is nonsingular, since it is one of the twenty-one 4-fold points of the arrangement $(K = 0)$, and the isotropy subgroup at the point is generated by the involutions associated with the four lines in $(K = 0)$ passing through it. The other two fixed points lie on the line corresponding to the involution h^2. So for each of the two points, we first divide a sufficiently small neighborhood by $\{1, h^2\}$ and then divide the quotient, which is smooth, by the involution h in $H/\{1, h^2\}$, which has now only an isolated fixed point. Note that there are forty-two points of this kind, which form again one $\mathrm{PSL}(2, F_7)$-orbit. We obtain thus one simple singular point of type A_1 on S. Note also that the forty-two points are on the curve $(\nabla = 0)$. The group $\mathrm{PSL}(2, F_7)$ has also exactly twenty-eight (cyclic) subgroups of order 3, which are conjugate to each other. Now let H be a subgroup of order 3 and h a generator of H. By the character table we know that h has eigenvalues 1, ω, ω^2 $(\omega^2 + \omega + 1 = 0)$, and that we have the corresponding eigenvectors v_0, v_1, v_2. Obviously v_0 and the other two are of different nature, although the corresponding points (v_1), (v_2), (v_0) on $P_2(\mathbb{C})$: (y_1, y_2, y_3) are the fixed points of h. The $\mathrm{PSL}(2, F_7)$-orbit passing through (v_0) consists of twenty-eight points which are exactly the 3-fold points of the arrangement $(K = 0)$. By the same reason as before, the image point on S of this orbit is nonsingular. On the other hand, (v_1) and (v_2) belong to one and the same orbit, whose image point on S is a simple singular point of type A_2. This orbit consists of fifty-six points which form the transversal intersection of $(f = 0)$ and $(C = 0)$. The group $\mathrm{PSL}(2, F_7)$ has exactly eight (cyclic) subgroups of order 7, which are also conjugate to each other. Now let H be a subgroup of order 7 and $h \in H$ a generator. The element h has the eigenvalues ζ, ζ^2, ζ^4 on V for some appropriate 7th root ζ of unity; h^2, h^4 have the same eigenvalues, while ζ^3, ζ^5, ζ^6 are the eigenvalues for h^3, h^5, h^6.

Let v_1, v_2, v_3 be the three eigenvectors of h, and (v_1), (v_2), (v_3) the corresponding fixed points on $P_2(\mathbb{C})$: (y_1, y_2, y_3). Now the normalizer, in which H is of index 3, permutes the three points (v_i) transitively, and thus, $3 \times 8 = 24$ fixed points of elements of order 7 form one $\mathrm{PSL}(2, F_7)$-orbit, whose image on S is a cyclic quotient singular point of type $(-2,-2,-3)$. (This means that the exceptional set in the resolution is a fan of nonsingular rational curves with self-intersection numbers $-2,-2,-3$.) These twenty-four points form the (transversal) intersection of the curves $(f=0)$ and $(\nabla=0)$, so they are exactly the inflection points of $(f=0)$. (The points (v_1), (v_2), (v_3) form an inflection-triangle "Wendedreieck" in the sense of [1] and there are eight such triangles.) These three quotient singular points form exactly the singular locus of S. To sum up, we obtain six rational curves on the minimal resolution $\hat{S}$ coming from the exceptional sets. We denote the (-2)-curve over the point of type A_1 by E' and the (-3)-curve, which is unique, by E''. The images of the curves $(f=0)$ and $(\nabla=0)$ are the exceptional curves of the first kind, when lifted to $\hat{S}$. Including these, we now have eight rational curves on $\hat{S}$ and we see immediately that there is a unique way of blowing down such that only E' and E'' survive afterwards. By computing the Euler number we conclude that this blowing down is $P_2(\mathbb{C})$. In fact we can identify it with $P_2(\mathbb{C})$: (ξ,η,ζ). Note that the curve $(\nabla=0)$ is contracted to the point $(\xi=\zeta=0)$ and the curve $(f=0)$ to the point $(\eta=\zeta=0)$. Note further that no curve in $P_2(\mathbb{C})$: (y_1,y_2,y_3) can be mapped onto the line $(\zeta=0)$ or the line $(\xi=0)$. In fact $(\zeta=0)$ is the image of E' and $\xi=0$ the image of E''. Now we ask what is the image on $P_2(\mathbb{C})$: (ξ,η,ζ) of the lifting to $\hat{S}$ of the image on S of the arrangement $(K=0)$. By relation (1.1), we see that this image is given by $H^*(\xi,\eta,\zeta)=0$ where we have put

$$\begin{aligned} H^*(\xi,\eta,\zeta) := {} & 2\xi\eta^3+(27\xi^2+63\xi\eta-22\eta^2)\xi\zeta \\ & -(469\xi-34\eta)\xi\zeta^2+2(43\xi-2\eta)\zeta^3-4\zeta^3. \end{aligned}$$

(By Boulanger's polynomial H [1, p. 424], $H^*(\xi,\eta,\zeta)=\zeta^4H(2\xi/\zeta,8\eta/\zeta)/512$.) To sum up, we have seen the following:

Proposition 1.1. *The group* $\mathrm{PSL}(2,F_7)$ *acts freely on*

$$\mathrm{Comp}^* := P_2(\mathbb{C})\colon (y_1,y_2,y_3)-\{(f=0)\cup(\nabla=0)\cup(K=0)\}.$$

By the mapping $(y_1,y_2,y_3)\mapsto(4\nabla^3, fC, 8f^3\nabla)$, *the quotient* $\mathrm{Comp}^*/\mathrm{PSL}(2,F_7)$ *is isomorphic to the complement of the three curves* $(\xi=0)$, $(\zeta=0)$, *and* $(H^*=0)$ *on* $P_2(\mathbb{C})$: (ξ,η,ζ).

To close this section we mention a diagram describing the interrelation of the spaces:

$$\begin{tikzcd} P_2(\mathbb{C})\colon (y_1,y_2,y_3) \arrow[d] & \hat{S} \arrow[dl, "\rho"'] \arrow[d, "\tau"] \\ S \arrow[r, "\varphi"] & P_2(\mathbb{C})\colon (\xi,\eta,\zeta) \end{tikzcd}$$

where π is the quotient map by $\mathrm{PSL}(2,F_7)$, ρ is the minimal desingularization, τ is the blowing down and φ is the map in the proposition.

2. The determination of the fundamental group

Recall that our objective is π_1 of the following complement of Klein's arrangement. We write

$$\mathrm{Comp} := (P_2(\mathbb{C}): (y_1, y_2, y_3)) - (K = 0).$$

However, the group PSL(2, F_7) does not operate freely on this, so we will compare this with the complement Comp* introduced in Proposition 1.1.

Proposition 2.1. *There is a natural isomorphism induced by the inclusion* Comp* $\subsetneq$ Comp:

$$\pi_1(\mathrm{Comp}^*, p_0) \cong \pi_1(\mathrm{Comp}, p_0) \times \mathbb{Z} \times \mathbb{Z}, \tag{2.1}$$

where p_0 is the common reference point for the two fundamental groups.

Proof. As is checked immediately, the three curves $(f = 0)$, $(\nabla = 0)$, $(K = 0)$ intersect transversally in their nonsingular points (they do not intersect at one point); a fortiori, the affine curves $(f = 0) - l$, $(\nabla = 0) - l$, $(K = 0) - l$ intersect transversally, where l is an arbitrarily fixed line in the arrangement $(K = 0)$. Moreover, we have $\mathrm{Comp} = \{P_2(\mathbb{C}) - l\} - \{(K = 0) - l\}$ and Comp* is the complement in $(P_2(\mathbb{C}) - l) \cong \mathbb{C}^2$ of the three affine curves. Since the curves $(f = 0)$, $(\nabla = 0)$ are nonsingular, we have the obvious isomorphisms:

$$\pi_1(P_2(\mathbb{C}) - \{l \cup (f = 0)\}, p_0) \cong \mathbb{Z},$$
$$\pi_1(P_2(\mathbb{C}) - \{l \cup (\nabla = 0)\}, p_0) \cong \mathbb{Z}.$$

Now the main theorem of [2] implies the isomorphism of the proposition.

We have thus seen that, for the determination of $\pi_1(\mathrm{Comp}, p_0)$, one need only compute $\pi_1(\mathrm{Comp}^*, p_0)$ and single out the two central elements which generate the direct component $\mathbb{Z} \times \mathbb{Z}$ in the right-hand side of (2.1).

Since we have seen in Proposition 1.1 that PSL(2, F_7) operates freely on the smaller complement Comp*, we have the natural exact sequence

$$\begin{aligned} 1 \to \pi_1(\mathrm{Comp}^*, p_0) &\to \pi_1(\mathrm{Comp}^*/\mathrm{PSL}(2, F_7), (p_0)) \\ &\to PSL(2, F_7) \to 1. \end{aligned} \tag{2.2}$$

We have already obtained a good description of Comp*/PSL(2, F_7), from which we are able to compute the middle term in (2.2) and also the homomorphism onto PSL(2, F_7).

Before computing π_1 of the quotient in Proposition 1.1 we still want to simplify the equation of the quartic curve $(H^* = 0)$ given in the preceding section. We make the following change of the homogeneous coordinates:

$$\bar{\xi} := \eta - 9\zeta, \qquad \bar{\eta} := \zeta, \qquad \bar{\zeta} := \xi + \eta - 7\zeta.$$

The inverse transformation is given by

$$\xi = -\bar{\xi} - 2\bar{\eta} + \bar{\zeta}, \qquad \eta = \bar{\xi} + 9\bar{\eta}, \qquad \zeta = \bar{\eta}.$$

Now the equation of the quartic curve is $\bar{H}=0$ where we have put

$$\bar{H}(\bar{\xi},\bar{\eta},\bar{\zeta}):=2\cdot\bar{\xi}^4-2\cdot\bar{\xi}^3\bar{\zeta}+13\cdot\bar{\xi}^2\bar{\eta}\bar{\zeta}+18\cdot\bar{\xi}\bar{\eta}\bar{\zeta}^2+64\cdot\bar{\eta}^2\bar{\zeta}^2-27\cdot\bar{\eta}\bar{\zeta}^3.$$

The lines $(\zeta=0)$ and $(\xi=0)$ are defined in the new coordinates by

$$\bar{\eta}=0,\qquad \bar{\xi}+2\bar{\eta}-\bar{\zeta}=0.$$

But now we will rather use the affine coordinates

$$x=\bar{\xi}/\bar{\zeta},\qquad y=\bar{\eta}/\bar{\zeta},$$

though, by this, the line at infinity is not entirely contained in Comp*. The equations of the above three curves are now

$$Q\colon G(x,y):=64y^2+(13x^2+18x-27)y+2x^3(x-1)=0,$$

$$L_1\colon y=0,$$

$$L_2\colon x+2y-1=0,$$

where we use Q, L_1, L_2 also to denote their zero sets. We cut these curves by the 1-parameter family of lines $x=\text{const}$. As we see immediately, the generic line in the family cuts the quartic curve Q transversally at two points. This implies that the point $(\bar{\xi}=\bar{\zeta}=0)$ at infinity is a double point of the curve $(\bar{H}=0)$, and in fact it is a tacnode (simple singular point of type A_3) of the curve, which corresponds to the 4-fold points of the arrangement $(K=0)$; we note that the line at infinity $\bar{\zeta}=0$ is the tangent at the tacnode. We are namely cutting the arrangement $(\bar{H}=0)\cup(\bar{\eta}=0)\cup(\bar{\xi}+2\bar{\eta}-\bar{\zeta}=0)$ by the pencil of lines passing through the tacnode. We now draw the real locus of the three curves (see Fig. 1). (Of course, Fig. 1 is not a

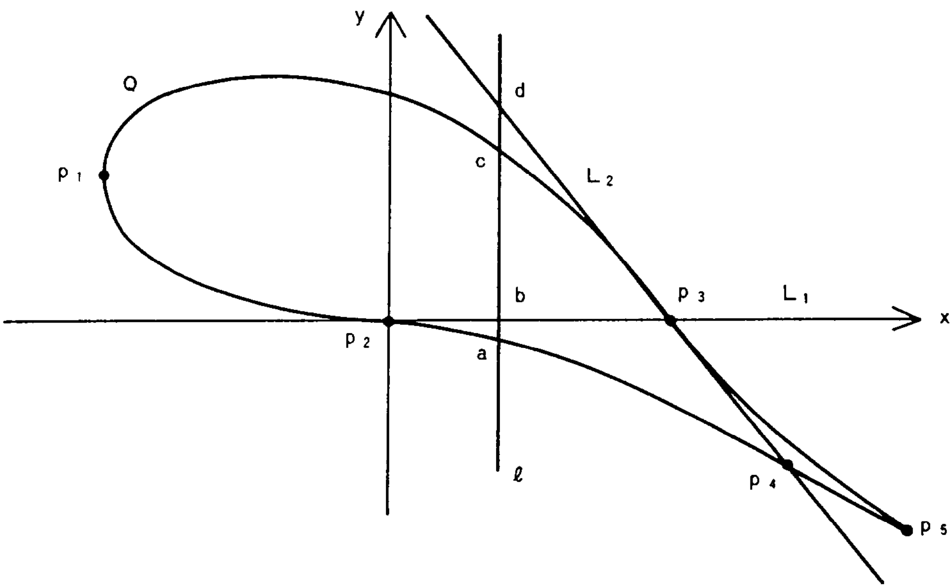

Fig. 1.

correct figure; we should have made some exaggeration to make the essential features clear.) As we observe in Fig. 1, the locus of Q in the real affine plane is compact (and connected); thus we also see that the tacnode of Q is isolated in $P_2(\mathbb{R})$. On the other hand the cusp of Q is clearly observed at the right-hand side of Fig. 1: this corresponds to the 3-fold points of the arrangement ($K=0$). The vertical line, denoted by l in Fig. 1, is the standard line fixed for the determination of π_1 of the complement of the arrangement $L_1 \cup L_2 \cup Q$. The intersection points of l with the arrangement are denoted from below by a, b, c, d, and e; e is thus the tacnode of Q at infinity. If we want a, b, c, d to be rational, we may put for example $l\colon x=5/11$; the points a, b, c, d have then the y-coordinates $4\cdot 121\cdot y=-3$, 0, 125, 132 respectively. We take a point p_0 on the complex line l for which the y-coordinate lies in the upper half-plane (e.g. $y=\sqrt{-1}$), and we draw the closed paths α, β, γ, δ on $l^* := l-\{a, b, c, d, e\}$ issuing from p_0, going straight toward a, b, c, d, surrounding them once counterclockwise and coming back to p_0 (see Fig. 2). We use the same letters to denote the classes of α, β, γ, δ in the fundamental group with the reference point p_0 of the complement of $Q \cup L_1 \cup L_2$. For brevity we set

$$G_0 := \pi_1(P_2(\mathbb{C}) - \{Q \cup L_1 \cup L_2\}, p_0).$$

Applying the Zariski–van Kampen method to Fig. 1, we obtain the following:

Lemma 2.2. *The group G_0 is generated by α, β, γ, δ. We have the relations among them*:

$$\gamma = (\alpha\beta)\alpha(\alpha\beta)^{-1}, \tag{R_1'}$$

$$(\alpha\beta)^3 = (\beta\alpha)^3, \tag{R_2'}$$

$$\beta(\gamma\delta) = (\gamma\delta)\beta, \tag{R_3'}$$

$$[\beta(\gamma\delta)^3, \delta] = 1, \tag{R_3''}$$

$$[\alpha, (\gamma\delta)^{-1}\delta(\gamma\delta)] = 1, \tag{R_4}$$

$$(\gamma\delta)\alpha(\gamma\delta) = (\delta\alpha)\gamma(\delta\alpha). \tag{R_5}$$

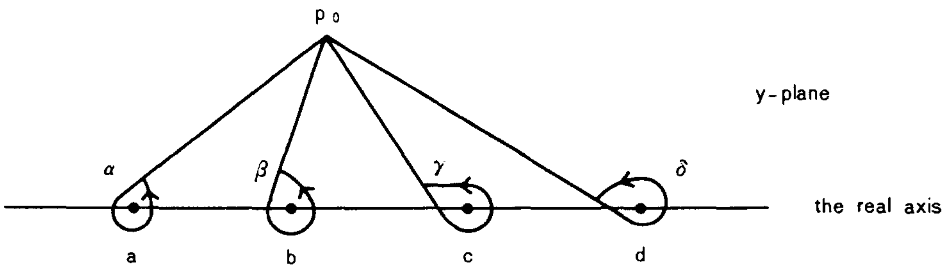

Fig. 2.

We obtain these relations by deforming the standard line l in the pencil of vertical lines $x = \text{const}$, avoiding those lines which do not intersect $Q \cup L_1 \cup L_2$ transversally and deforming the paths α, β, γ, δ simultaneously. The relations are then the identities between the original homotopy classes and their deformed classes, when we come back to the original position of l in all possible ways. We proceed mostly along the real axis of the parameters c in the pencil $x = c$. Whenever we are confronted with a singular vertical line, we turn it counterclockwise within a small complex neighborhood of the singular value of c by the angle π or 2π according as we want to pass by it or to surround it once. This way of deformation is sufficient in this case, since the parameter c takes only real values for the singular vertical lines. Moreover, there lies exactly one multiple intersection point with the arrangement $Q \cup L_1 \cup L_2$ (except the tacnode e) for each of the singular lines. The points p_1, p_2, p_3, p_4, p_5 in Fig. 1 are the multiple points in this sense. From p_2 we obtain (R_2') in the usual way and from p_3 the relations (R_3') and (R_3''). After passing by p_2 the class α should be replaced by $(\beta\alpha\beta)^{-1}\alpha(\beta\alpha\beta) = (\alpha\beta)\alpha(\alpha\beta)^{-1}$, while γ is unchanged. Thus, from p_1, we obtain the identity (R_1'). Now, after passing by p_3, the class δ should be replaced by $(\gamma\delta)^{-1}\delta(\gamma\delta)$, and γ by $\bar{\gamma} := (\delta\gamma\delta)^{-1}\gamma(\delta\gamma\delta)$. Thus, from p_4, we obtain (R_4) and from p_5 the relation $\alpha\bar{\gamma}\alpha = \bar{\gamma}\alpha\bar{\gamma}$, which is equivalent to (R_5) under (R_4).

The relations in Lemma 2.2 are not yet sufficient to define G_0 as an abstract group, since the line at infinity $\bar{\zeta} = 0$ (corresponding to $x = \infty$) is not contained entirely in the arrangement. Moreover we have the tacnode of Q on this line, whose isolatedness in the real locus provides a difficulty to overcome. We will now blow up $P_2(\mathbb{C})$: $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ at $e = (\bar{\xi} = \bar{\zeta} = 0)$. By the pencil of lines passing through e, almost all of which we have observed above, the blowing up is the $P_1(\mathbb{C})$-bundle Σ_1 over $P_1(\mathbb{C})$. This is covered by two patches, each isomorphic to the trivial bundle $P_1(\mathbb{C}) \times \mathbb{C}$, and one of which we can consider observed above. To introduce the coordinates for another patch, we set now

$$\tilde{x} := \frac{1}{x} = \frac{\bar{\zeta}}{\bar{\xi}}, \qquad \tilde{y} := \frac{x}{y} = \frac{\bar{\xi}}{\bar{\eta}},$$

where $\tilde{y}$ is the affine coordinate for $P_1(\mathbb{C})$ and $\tilde{x}$ that for $\mathbb{C}$ in the patch. Now, in the $(\tilde{x}, \tilde{y})$-plane, the line at infinity is the strict transform of L_1, and the transforms of Q and L_2 are given by

$$Q\colon 2(\tilde{x} - 1)\tilde{y}^2 + \tilde{x}(27\tilde{x}^2 - 18\tilde{x} - 13)\tilde{y} - 64\tilde{x}^2 = 0,$$

$$L_2\colon (\tilde{x} - 1)\tilde{y} - 2 = 0.$$

The exceptional curve of the blowing up is given by

$$E\colon \tilde{y} = 0$$

in this patch. We should now think that E is the line of infinity in the (x, y)-plane, which was identified with the tangent line at the tacnode. The strict transform of this tangent is now viewed to be the vertical line $\tilde{x} = 0$. The tacnode becomes an ordinary node, when Q is lifted to the blowing up. To obtain the remaining relations we will still move in the same pencil $\tilde{x}=c$ along a sufficiently small neighborhood of the real axis of the c-plane. Since the node is isolated in the real locus of the curve Q, we have the imaginary $\tilde{y}$-roots of the equation as $\tilde{x}$ stays in the interval $(-1,0)$ or $(0,7/9)$. Thus we have to trace how the complex $\tilde{y}$-roots move as $\tilde{x}$ moves from -1 to $7/9$. To obtain the equation of this locus in the complex $\tilde{y}$-plane, we set $\tilde{y}=a+b\mathrm{i}$ with real variables a, b in the equation of Q. We then obtain the equation of the form $R(a,b,\tilde{x})+\mathrm{i}bI(a,b,\tilde{x})=0$ where $\tilde{x}$ is assumed to be real, R and I are explicitly given by

$$R-aI=\{(a^2+b^2)(\tilde{x}-1)+32\tilde{x}^2\}\times 2,$$

$$-I=4a(\tilde{x}-1)+\tilde{x}(27\tilde{x}^2-18\tilde{x}-13).$$

By taking the resultant of these polynomials with respect to $\tilde{x}$, we obtain the sextic equation for the locus in the real (a,b)-plane, which we do not write down here but we actually see by some simple computer graphics that the locus is compact and has the simple shape as pointed out in Fig. 3. The arrows in Fig. 3 indicate the direction in which the conjugate pair of $\tilde{y}$-roots moves as $\tilde{x}$ moves from -1 to $7/9$. We also give a picture for the real loci of the curves Q, L_2, E (see Fig. 4), where the chain curve denotes the locus of the real part a as $\tilde{x}$ moves on $(-1,7/9)$. These figures give us sufficient informations to get the remaining relations. For example we see the following: The two $\tilde{y}$-roots on the line $\tilde{x}=-\varepsilon$ ($\varepsilon>0$; sufficiently small) are near the origin $\tilde{y}=0$ which is the intersection of the line with E, and they are situated as in Fig. 5, where $\tilde{p}_0$ denotes a point in the upper half plane obtained by

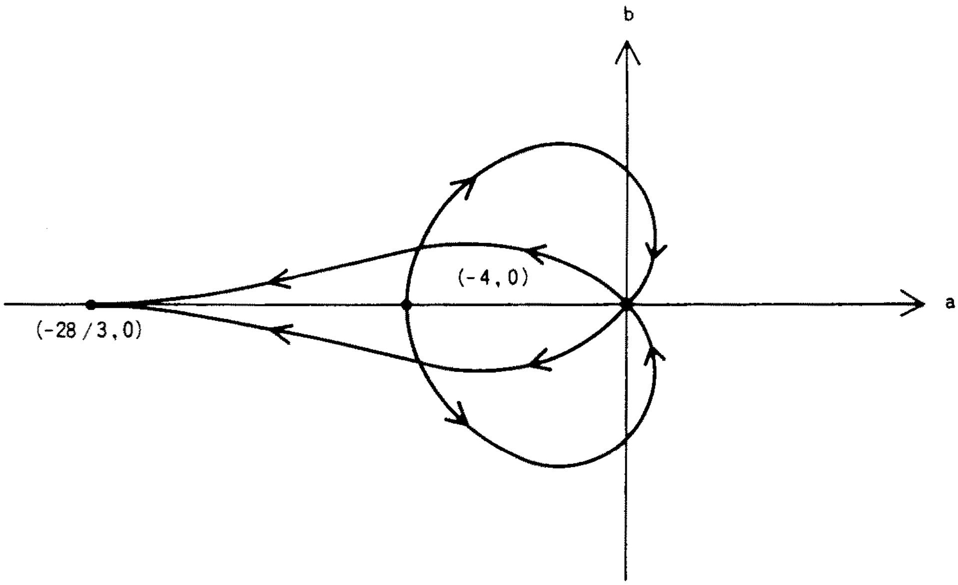

Fig. 3.

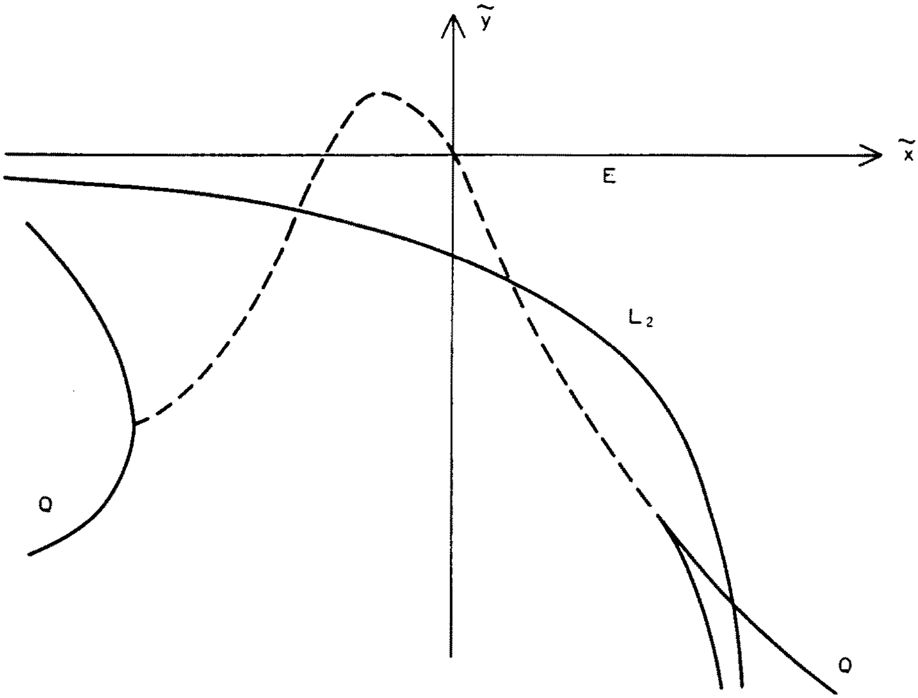

Fig. 4.

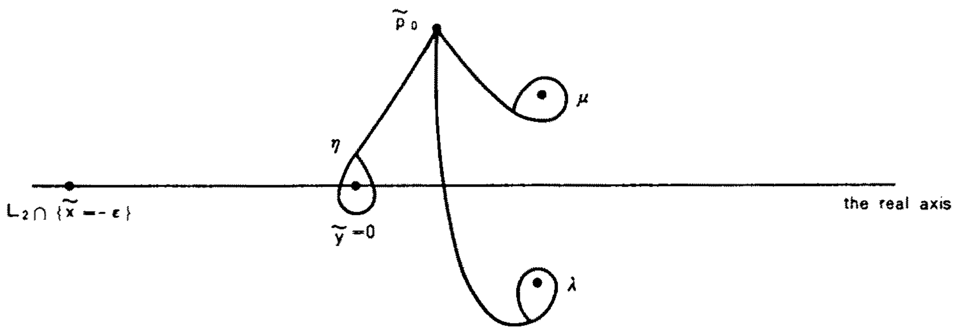

Fig. 5.

moving p_0 from $\tilde{x} = -1$ without touching the arrangement $Q \cup L_1 \cup L_2$. Let η, λ, μ be the homotopy classes represented by the paths in Fig. 5. We consider them to be elements of G_0. Then, by the continuous deformation of paths, we obtain the following identities:

$$\eta = \alpha\beta\alpha(\alpha\beta\alpha\gamma\delta\alpha\beta)^{-1},$$

$$\lambda = (\alpha\beta\alpha)\gamma(\alpha\beta\alpha)^{-1},$$

$$\mu = \gamma.$$

For the deduction of these we have of course used some relations of Lemma 2.2. By turning once around the singular line $\tilde{x}=0$, we further deduce that $\eta\lambda\mu=\lambda\mu\eta=\mu\eta\lambda$, i.e., that this element commutes with η, λ, μ and thus with any of their products. Since $\eta\lambda\mu=\alpha\beta\alpha(\delta\alpha\beta)^{-1}(\alpha\beta)^{-1}=\gamma(\alpha\beta\delta)^{-1}$, we see that $\gamma=\alpha\beta\alpha(\alpha\beta)^{-1}$ and $\alpha\beta\delta$ commute, which implies the important relation:

$$\alpha\delta=\delta\gamma. \tag{R_1}$$

Now we should examine, whether we had all relations for the determination of the fundamental group. We were operating in the two patches of the blowing up. We denote the patch in which we have the coordinates (x, y) by U_1 and that in which we have $(\tilde{x}, \tilde{y})$ by U_2. By U_1^*, U_2^* we denote the complements of the arrangement in U_1, U_2. What we have done above is the determination of $\pi_1(U_i^*, p_0)$ $(i=1,2)$, their common generators α, β, γ, δ being obviously taken from $\pi_1(U_1^*\cap U_2^*, p_0)$. However, the last group is not generated by α, β, γ, δ only. For, $U_1\cap U_2$ is a $P_1(\mathbb{C})$-bundle over $\mathbb{C}^*$ and π_1 of $\mathbb{C}^*$ is nontrivial. The group $\pi_1(U_1^*\cap U_2^*, p_0)$ is generated by α, β, γ, δ and a homotopy class which is mapped to a generator of $\pi_1(\mathbb{C}^*)\cong\mathbb{Z}$ and the van Kampen theorem asserts that the only remaining relation is obtained by equalizing the two expressions by α, β, γ, δ of this class in the two groups $\pi_1(U_i^*, p_0)$ $(i=1,2)$. We further note that this last relation comes essentially from the bundle structure, i.e., from the blowing up and that it is thus a local relation near the center of the blowing up. The curve Q has the two irreducible branches at the tacnode e and the above homotopy classes λ, μ are those surrounding them. Since they generate the local fundamental group at e, the class η, which surrounds the exceptional divisor should be a product of λ, μ. In fact we have $\eta=\lambda\mu$, from which we deduce by using the other relations:

$$\delta\alpha\beta\gamma\alpha\gamma=1. \tag{R_2}$$

Recall that we have used $[\gamma, \alpha\beta\delta]=1$ to deduce (R_1) from (R_1'). By the conjugation $x\mapsto(\alpha\beta)^{-1}x(\alpha\beta)$ and (R_1'), this is equivalent to $[\alpha, \delta\alpha\beta]=1$ which we can also derive from (R_1') and (R_1). This together with (R_2) implies $[\alpha, \gamma\alpha\gamma]=1$; that is, the following:

$$\alpha\gamma\alpha\gamma=\gamma\alpha\gamma\alpha. \tag{R_3}$$

Since $\beta^{-1}=\gamma\alpha\gamma\delta\alpha$ by (R_2), we can deduce (R_3') from (R_1) and (R_5). In the next section we will see that (R_2') and (R_3'') are also derived from (R_1)-(R_5). Thus we obtain

Proposition 2.3. *The group* $G_0=\pi_1$ $(\mathrm{Comp}^*/\mathrm{PSL}(2, F_7), p_0)$ *is generated by* α, β, γ, δ *with the fundamental relations* (R_1), (R_2), (R_3), (R_4), (R_5).

3. The central elements and the homomorphism

To determine the two central elements corresponding to the direct factor $\mathbb{Z}\times\mathbb{Z}$ in Proposition 2.1, we first mention the scheme of the blowing down $\hat{S}\to P_2(\mathbb{C})$:

(ξ, η, ζ) which we did not describe explicitly in Section 1. Recall that $\hat{S}$ is the minimal resolution of $S = P_2(\mathbb{C})/\mathrm{PSL}(2, F_7)$ and that S has the three quotient singular points of types (-2), $(-2, -2)$, $(-2, -2, -3)$ which give rise to the six rational curves on $\hat{S}$. By F and D we denote the strict transform to $\hat{S}$ of the image on S of the curves $(f = 0)$ and $(\nabla = 0)$. We have the graph of these eight curves in Fig. 6. The thick lines here denote the curves which remain after the blowing down and the chain indicates a part near the eight rational curves of the strict transform of the image on S of $(K = 0)$. We see immediately from the above scheme that the homotopy class surrounding D once corresponds to the generator $\beta(\gamma\delta)^3$ of the center for the local fundamental group at p_3 (see Section 2) and that the class surrounding F once corresponds to $(\alpha\beta)^3$ which is also a generating central element of the local π_1 at p_2. By Proposition 2.1 we know that these belong to the center of G_0, which should however be derived from the relations (R_i) $(i = 1, 2, \ldots, 5)$. We will in fact give a purely algebraic proof for this. We need several lemmas.

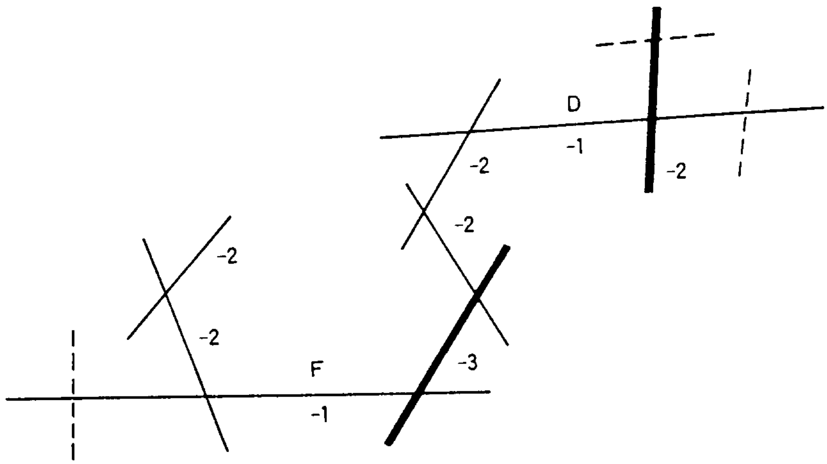

Fig. 6.

Lemma 3.1. *The group G_0 is generated by α and β.*

Proof. The relations (R_2) and (R_3) imply that $\delta\alpha\beta$ commutes with α, which implies the identity $(\alpha\beta)\alpha(\alpha\beta)^{-1} = \delta^{-1}\alpha\delta = \gamma$ where we have used (R_1) in the last identity. By (R_2) we see that δ is also expressed by α and β. $\square$

In view of this lemma $(\alpha\beta)^3$ is central if it commutes with α. Since (R_2) implies $\alpha\beta = (\gamma\alpha\gamma\delta)^{-1}$, this is proved by the following:

Lemma 3.2. *$(\gamma\alpha\gamma\delta)^3$ commutes with α.*

Proof. $(\gamma\alpha\gamma\delta)^3\alpha = \gamma\alpha\gamma(\delta\gamma)\alpha\gamma(\delta\gamma)\alpha\gamma\delta\alpha$ which is by (R_1) equal to $(\gamma\alpha\gamma\alpha)\delta\alpha\gamma\alpha(\delta\alpha\gamma\delta\alpha)$ which is by (R_3), (R_5) equal to $\alpha\gamma\alpha\gamma\delta(\alpha\gamma\alpha\gamma)\delta\alpha\gamma\delta = \alpha\gamma\alpha\gamma\delta\gamma\alpha\gamma(\alpha\delta)\alpha\gamma\delta = \alpha(\gamma\alpha\gamma\delta)^3$. $\square$

To prove the assertion for $\beta(\gamma\delta)^3$ we begin with the deduction of (R_3') in Section 2.

Lemma 3.3. *The elements β and $\gamma\delta$ commute.*

Proof. Since $\beta=(\gamma\alpha\gamma\delta\alpha)^{-1}$ by (R_2), it suffices to prove the commutativity of $\gamma\delta$ and $\gamma\alpha\gamma\delta\alpha$. But we can transform the product as follows: $\gamma\delta(\gamma\alpha\gamma\delta\alpha)=\gamma(\delta\gamma)\alpha\gamma\delta\alpha=\gamma\alpha(\delta\alpha\gamma\delta\alpha)=\gamma\alpha(\gamma\delta\alpha\gamma\delta)=(\gamma\alpha\gamma\delta\alpha)\gamma\delta$, which was to be proved. □

Lemma 3.4. $\beta(\gamma\delta)^3=\alpha^{-1}(\gamma\delta)^{-1}\delta^2(\gamma\delta)=\alpha^{-1}\{(\gamma\delta)^{-1}\delta(\gamma\delta)\}^2$.

Proof. We need only prove the first identity. But this is equivalent to $\gamma\delta\alpha\beta\gamma\delta\gamma=\delta$. By (R_2) we have $\gamma\delta\alpha\beta\gamma=\alpha^{-1}$, so this follows from (R_1). □

Now we summarize these results in the following:

Proposition 3.5. *$(\alpha\beta)^3$ and $\beta(\gamma\delta)^3$ are central elements of G_0.*

Proof. It suffices to prove the assertion for the second element. By Lemma 3.3 we see that this commutes with both β and $\gamma\delta$. By Lemma 3.4 and (R_4) it also commutes with α. Since $\gamma=(\alpha\gamma\delta\alpha\beta)^{-1}$, it commutes with γ. These obviously imply that $\beta(\gamma\delta)^3$ is central. □

Here we want to remark that the group G_0 is generated by α and δ. For, we have $\gamma=\delta^{-1}\alpha\delta$ by (R_1), so we see by (R_2) that β is also expressed by α and δ.

Now, since our objective is to compute $\pi_1(\text{Comp}, p_0)$, we kill the central elements in Proposition 3.5 by putting the following two relations:

$$(\alpha\beta)^3=1, \qquad (R_6)$$

$$\beta(\gamma\delta)^3=1. \qquad (R_7)$$

We denote the group generated by α, β, γ, δ with the fundamental relations (R_i) $i=1,2,\ldots,7$ by G. By Lemma 3.4, (R_7) is equivalent to $\alpha=(\gamma\delta)^{-1}\delta^2(\gamma\delta)$. Since $\gamma=\delta^{-1}\alpha\delta$, we obtain the following pseudo-braid relation:

$$\alpha\delta^2\alpha=\delta^2\alpha\delta^2. \qquad (P)$$

From this and (R_1) we deduce

$$\gamma\delta\alpha=\delta\alpha\delta^2. \qquad (3.1)$$

We also make use of the identity

$$\delta\gamma\alpha=\alpha\delta\alpha, \qquad (3.2)$$

which follows immediately from (R_1). Since $(\alpha\beta)^{-1} = \gamma\alpha\gamma\delta$, we can transform the relation (R_6) by using (3.1), (3.2), etc. as follows:

$$\begin{aligned} 1 &= \gamma\alpha\gamma\delta\gamma\alpha\gamma(\delta\gamma)\alpha\gamma\delta = \delta\gamma\alpha\gamma\delta(\gamma\alpha\gamma\alpha)\delta\alpha\gamma \\ &= (\delta\gamma\alpha)(\gamma\delta\alpha)\gamma\alpha(\gamma\delta\alpha)\gamma = \alpha\delta\alpha\cdot\delta\alpha\delta^2\cdot\gamma\alpha\cdot\delta\alpha\delta^2\gamma \\ &= \delta\alpha\delta\alpha\delta(\delta\gamma\alpha)(\delta\alpha\delta)(\delta\gamma\alpha) = \delta\alpha\delta\alpha\delta\cdot\alpha\delta\alpha\cdot\delta\alpha\delta\cdot\alpha\delta\alpha = (\delta\alpha)^7. \end{aligned}$$

In this process, we have sometimes transfered the bottom term to the top and vice versa in the product which is equal to 1. Anyway we have obtained a reformulation of (R_6):

$$(\alpha\delta)^7 = 1. \tag{Q}$$

Since $\gamma = \delta^{-1}\alpha\delta$, the following is a direct reformulation of (R_3).

$$[\alpha, (\delta^{-1}\alpha\delta)\alpha(\delta^{-1}\alpha\delta)] = 1. \tag{R}$$

Proposition 3.6. *The group $G = G_0/(\mathbb{Z}\times\mathbb{Z})$ is generated by α and δ with the fundamental relations (P), (Q) and (R).*

The relation (R_1) is regarded as the definition of γ and (R_2) as that of β. (R_3) is exactly (R). However we leave to the reader the deduction of the other relations from (P), (Q), (R).

From Proposition 2.1 we know that there must be a homomorphism of G onto PSL(2, F_7) whose kernel is isomorphic to the desired group $\pi_1(\mathrm{Comp}, p_0)$. Now let us determine it. Note that PSL(2, F_7) is obtained as a factor group of G by adding sufficiently many relations to (P), (Q), (R). Recall that the homotopy class α is the one surrounding once the quartic curve Q which is the image of the arrangement $(K=0)$. Since each line in $(K=0)$ is the one-dimensional component of the fixed point set of an involution in PSL(2, F_7), we have to put at least the following relation:

$$\alpha^2 = 1. \tag{S}$$

This together with (P), (Q), (R) defines a factor group of G, which we denote by G_1. By (P) and (S), we obtain $\delta^4 = 1$ for G_1. Thus every element of G_1 can be expressed to be a product in which α and a power of the form δ^i $(1 \leq i \leq 3)$ appear alternatively, which we call a word. Then we can prove that every word of length ≥ 11 can be reduced to a shorter word by using (P), (Q), (R), (S). Thus G_1 is a finite group. By a cumbersome enumeration we see that G_1 is isomorphic to PSL(2, F_7). We omit the details, since this is too long and even not practical. We will rather point out that the homomorphism of G_1 onto PSL(2, F_7) is unique up to an automorphism of the latter group because of the relations $\alpha^2 = \delta^4 = (\alpha\delta)^7 = 1$. Explicitly the desired homomorphism h of G onto PSL(2, F_7) is given by

$$h(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad h(\delta) = \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}. \tag{3.3}$$

To sum up, we have proved:

Theorem 3.7. *We have the exact sequence*

$$1 \to \pi_1(\mathrm{Comp}, p_0) \to G \xrightarrow{h} \mathrm{PSL}(2, F_7) \to 1$$

where Comp *is the complement of Klein's arrangement of twenty-one lines on* $P_2(\mathbb{C})$, *G the group generated by* α, δ *with the relations* (P), (Q), (R) *and h the homomorphism given by* (3.3).

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